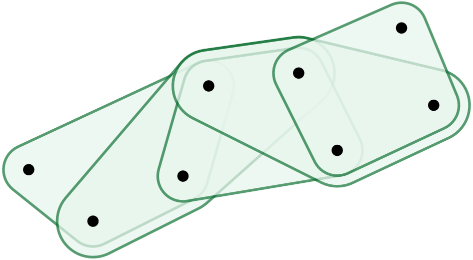
Spanning tight components in 4-uniform hypergraphs



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London School of Economics, United Kingdom

British Combinatorial Conference, 10 July 2026

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Example

- Dirac: Minimum degree $n/2 \implies$ Hamiltonicity.

Spanning subgraphs and tight connectivity

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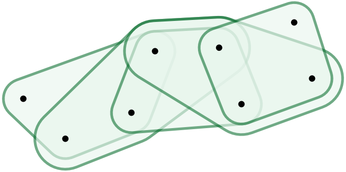

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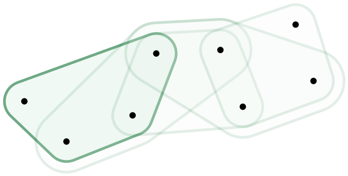

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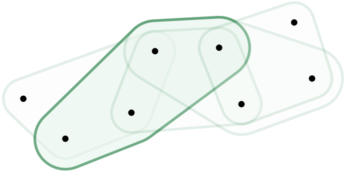

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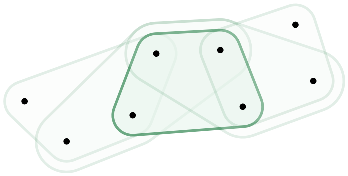

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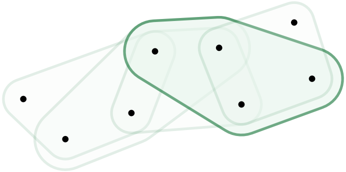

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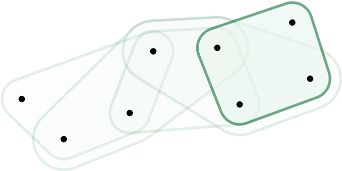

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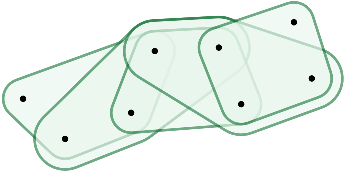

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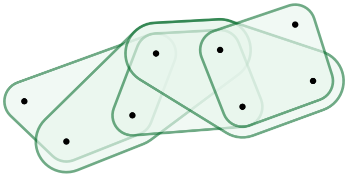
- A tight component is an edge-maximal tightly connected subgraph.



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Spheres in hypergraphs

“Definition”

A k-sphere in a k-graph is a tightly connected subgraph “homeomorphic” to a $(k-1)$-sphere.

Example: a spanning 1-sphere in a graph is a Hamilton cycle.

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Question (Conlon, Gowers)

What degree conditions force a spanning $(k-1)$-sphere in a k-graph?

Codegree

Definition

The codegree of a $(k-1)$-tuple T is the number of edges containing T.

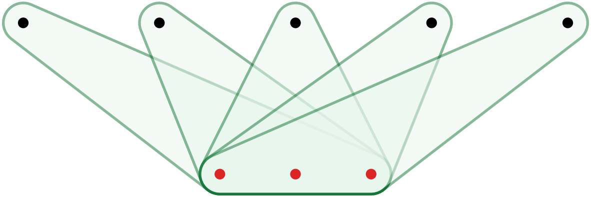

Figure: A triple of codegree 5.

Minimum codegree and spanning spheres

Theorem (Georgakopoulos, Haslegrave, Montgomery, Narayanan (2022))

Every n-vertex 3-graph with minimum codegree at least $n/3 + o(n)$ contains a spanning 2-sphere.

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Remark

GHMN constructed k-graphs with

- *minimum codegree $\approx n/k$,*
- *no spanning $(k-1)$-spheres,*
- *no spanning tight components.*

Minimum codegree and spanning tight components

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- $k = 2$: *trivial.*
- $k = 3$: *a short argument (Cooley, Mycroft).*

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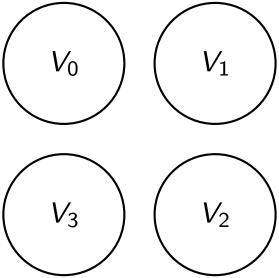
Theorem (Di Braccio, H., Lada, Neve, Zhang (2026+))

Every n-vertex 4-graph with minimum codegree at least $\lfloor n/4 \rfloor$ has a spanning tight component.

$\lfloor n/4 \rfloor$ is tight

Construction (Georgakopoulos, Haslegrave, Montgomery, Narayanan (2022))

Equipartition $V = V_0 \sqcup V_1 \sqcup V_2 \sqcup V_3$.
For each $v \in V_i$*, set* $v' := i$.



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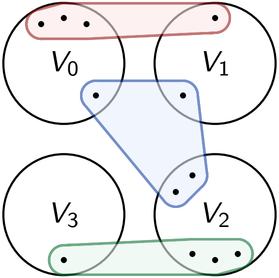
- $a' + b' + c' + d' \equiv 1 \pmod 4$



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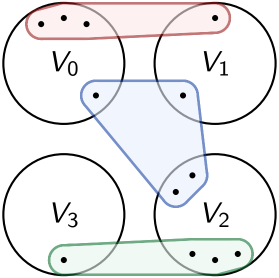

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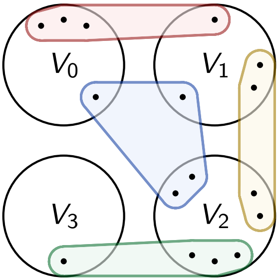
- † $a' \equiv b' \equiv c' - 1 \equiv d' - 1 \pmod 4$.



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Proof outline

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- *If any vertex misses two colours, then* *merge* *these colours.*

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Outline of the proof

Show that

- *there are ≤ 6 colours,*
- *if there is no spanning tight component, then there are ≥ 7 colours.*

There are ≤ 6 colours

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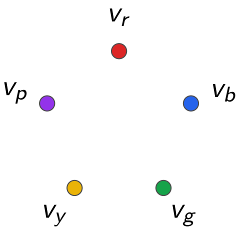
There are ≤ 5 colours in $L(u)$.
Since u misses at most 1 colour, there are ≤ 6 colours in $K_n^{(3)}$.

There are ≥ 7 colours

For each colour c, let v_c avoid colour c.



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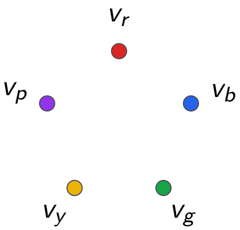

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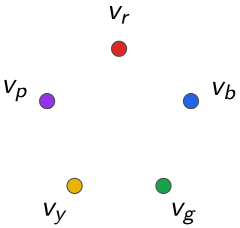

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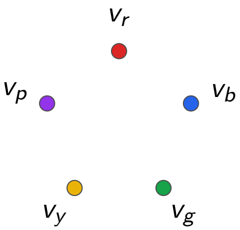
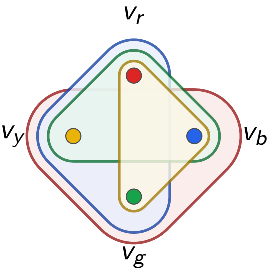


Let T be a minimal subset of $\{v_c \mid c \text{ is a colour}\}$ with no spanning colour.
Step 1: Understand the structure of T.
Step 2: Derive a contradiction using the minimum degree condition.

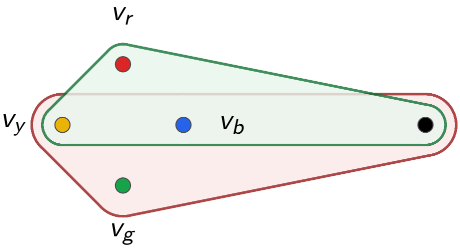
Example: ruling out 4 colours (step 1)

If $|T| = 4$, we get a rainbow $K_4^{(3)}$.



Example: ruling out 4 colours (step 2)

Each triple is contained in $\geq \lfloor n/4 \rfloor$ edges.
By pigeonhole, two triples 'extend' to the same vertex.



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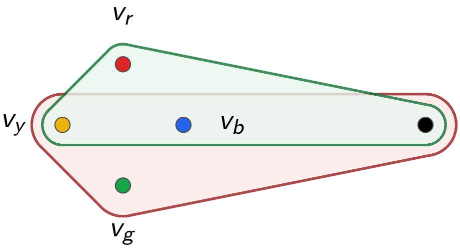
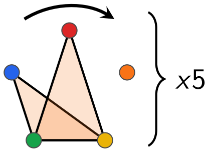


There are two tightly-connected edges with different colours. $\lightning$

Ruling out 5 colours, 6 colours . . .

If $|T| = 5$:

- all colours appear 2 times.



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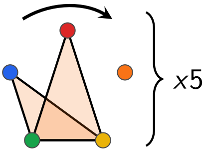

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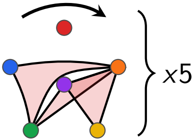

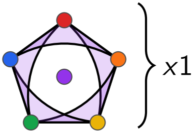

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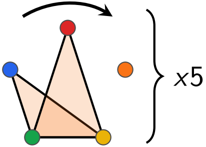

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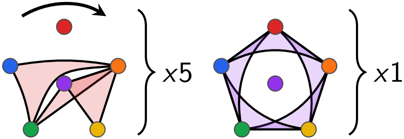

$\longrightarrow$ . . . can use these structures to derive a contradiction. $\square$

Open problems

Conjecture (GHMN (2022), ILMPS (2025))

Every n-vertex k-graph with minimum codegree at least n/k contains a

- *spanning tight component (ILMPS, 2025),*
- *spanning $(k-1)$-sphere (GHMN, 2022).*

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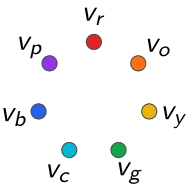

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Question (from upper regime)

Can we bound the number of colours in an edge-colouring of K_n where every edge lies in at least $f(n)$ monochromatic triangles?