Reconstructing Graphs from their Recolouring Graphs

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What is a k-colouring?

Definition

A k-colouring of a graph G is an assignment of **at most** k colours to the vertices of G, where no two adjacent vertices are assigned the same colour.

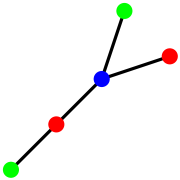
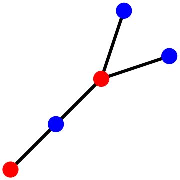



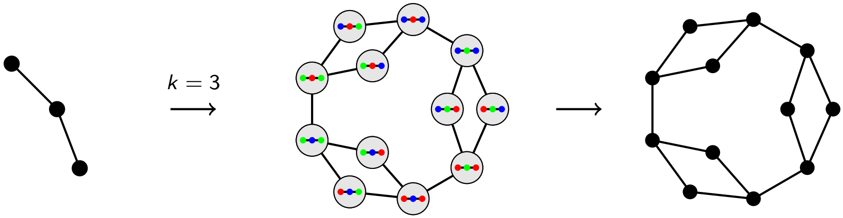
Figure: Some 3-colourings of a graph.

What is a recolouring graph?

Definition

Let G be a graph. Let $k \in \mathbb{N}$.

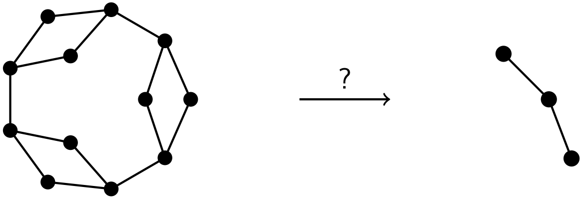
The k-recolouring graph $\mathcal{C}_k(G)$ of G is the graph whose vertex set is the set of k-colourings of G, with an edge between two k-colourings if and only if they differ at exactly one vertex.



Can we reconstruct G from its recolouring graphs?

Question

Given a non-empty recolouring graph $\mathcal{C}_k(G)$ (where k and G are unknown), can we reconstruct G?



Recent history

Conjecture (Asgarli, Krehbiel, Levinson, Russell (2024))

Every graph G is uniquely determined by the collection $\{\mathcal{C}_k(G)\}_{k\in\mathbb{N}}$.

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Theorem (Hogan, Scott, Tamitegama, Tan (2024))

Let G be a graph on n vertices. Given $k > 5n^2$, we can determine G from $\mathcal{C}_k(G)$.

Our result

Definition

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$\mathcal{C}_k(G)$ *is empty (has no vertices) if* $k < \chi(G)$.

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Theorem (Berthe, Brosse, H., van den Heuvel, Hoppenot, Pierron (2025))

Let G be a graph. We can determine G from its k-recolouring graph $\mathcal{C}_k(G)$ for any $k > \chi(G)$.

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Remark

The proof uses many of the ideas from the Hogan, Scott, Tamitegama, and Tan proof.

What if $k = \chi(G)$?

Observation

If $k = \chi(G)$, it is **not possible** *in general to determine G from $\mathcal{C}_k(G)$.*

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Examples

Any connected bipartite graph has the same 2-recolouring graph.

Any k-tree has the same $(k+1)$-recolouring graph.

...and so on.

Outline of the proof

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1. *Fix a colouring $c \in \mathcal{C}_k(G)$.*
2. *Use the* **local structure** *near c to construct a* **candidate graph** G_c.
3. *Show that the largest of these candidate graphs is isomorphic to G.*

Detecting vertices

Observations

Each **edge** *of* $\mathcal{C}_k(G)$ *corresponds to a* **vertex** *of* G.

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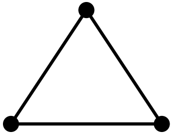
Each **clique** *in* $\mathcal{C}_k(G)$ *corresponds to a* **vertex** *of* G.



Detecting vertices

Fix some colouring $c \in \mathcal{C}_k(G)$.

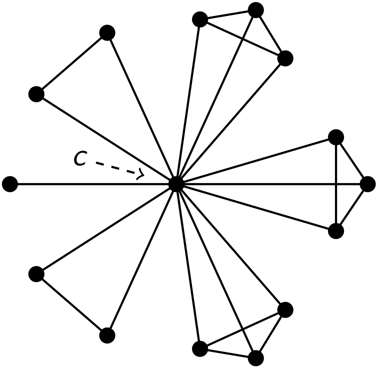


Figure: A colouring $c \in \mathcal{C}_k(G)$ and the maximal cliques containing c.

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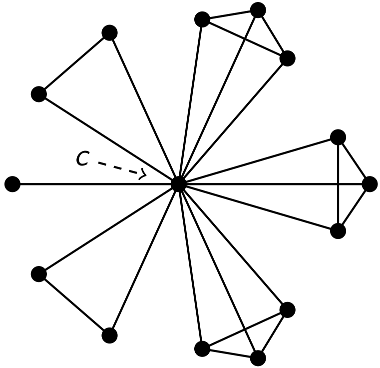

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Observations

Each maximal clique containing c corresponds to some vertex of G.

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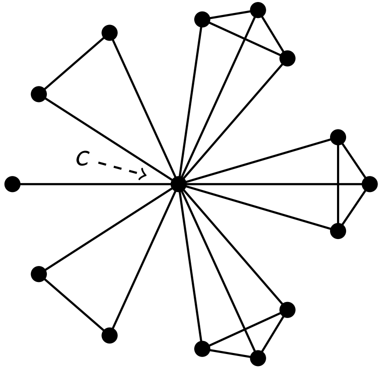

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Each maximal clique containing c corresponds to some vertex of G.

Different *cliques correspond to* **different** *vertices of G.*

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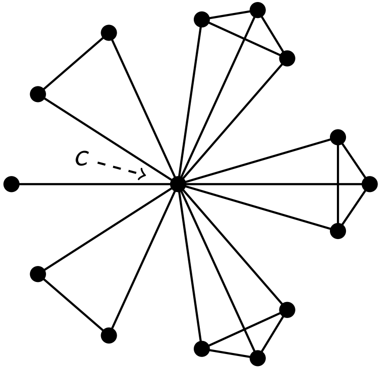

Figure: A colouring $c \in \mathcal{C}_k(G)$ and the maximal cliques containing c.

Observations

Each maximal clique containing c corresponds to some vertex of G.

Different *cliques correspond to* **different** *vertices of G.*

So every colouring $c \in \mathcal{C}_k(G)$ is in **at most** *n maximal cliques.*

(n is the number of vertices of G).

Detecting vertices

Fix some colouring $c \in \mathcal{C}_k(G)$.

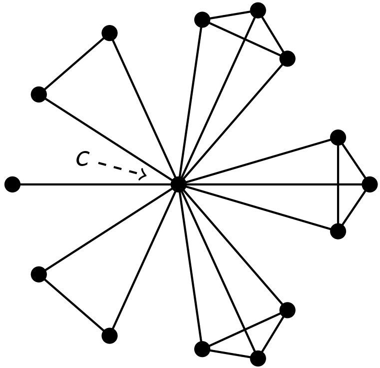

Figure: A colouring $c \in \mathcal{C}_k(G)$ and the maximal cliques containing c.

Observations (Continued)

Any colouring which doesn't use every colour is in **exactly** n *maximal cliques.*

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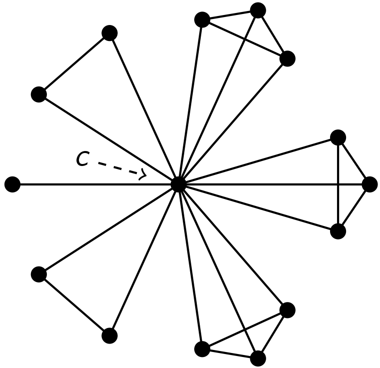


Figure: A colouring $c \in \mathcal{C}_k(G)$ and the maximal cliques containing c.

Observations (Continued)

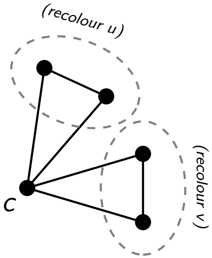
Any colouring which doesn't use every colour is in **exactly** *n maximal cliques.*

If $k > \chi(G)$, there are colourings which don't use every colour, so we can determine n from $\mathcal{C}_k(G)$.

(Count the number of maximal cliques each colouring is in, and take the maximum).

What about edges?

Fix $c \in \mathcal{C}_k(G)$ and focus on two cliques containing c, corresponding to two vertices u and v.



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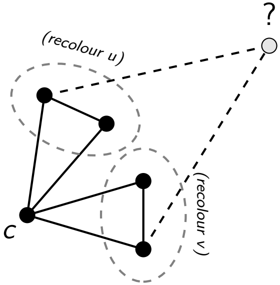
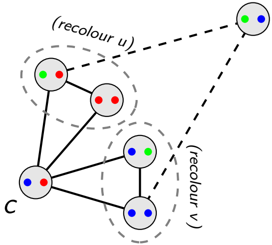


Then look at **common neighbours** of these cliques.

What about edges?

If uv is **not** an edge, then every pair closes to a 4-cycle.



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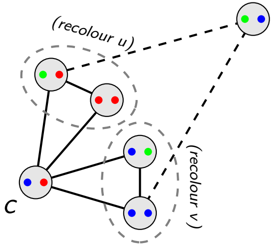

Main idea of the proof

For each $c \in \mathcal{C}_k(G)$, build a **candidate graph** *G_c as follows:*

- *Look at the cliques $C_{v_a}, \ldots, C_{v_p}$ containing c.*
- *Add an edge uv whenever we find a pair $c_u \in C_u$ and $c_v \in C_v$ which does* **not** *close to a 4-cycle.*

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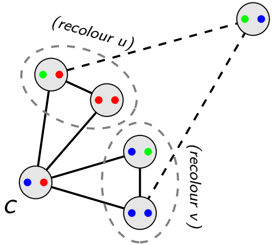

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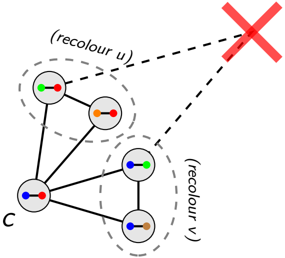
- *Add an edge uv whenever we find a pair $c_u \in C_u$ and $c_v \in C_v$ which does* **not** *close to a 4-cycle.*

Each candidate graph G_c is a **subgraph** *of G.*

What about edges?

If there is a colour **not** used by c, then if uv **is** an edge, then some pairs do **not** close to a 4-cycle.



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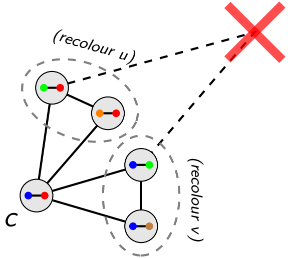

Observations

In this case, the candidate graph G_c is **isomorphic** *to G.*

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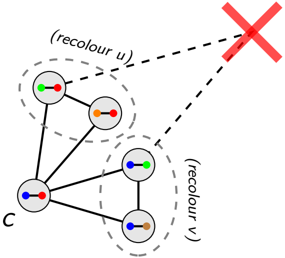

Observations

In this case, the candidate graph G_c is **isomorphic** *to G.*

Hence if $k > \chi(G)$, any candidate graph with a maximum number of vertices and edges is isomorphic to G. (This completes the proof).

Swapping multiple colours

Question

What if we instead allow Kempe swaps?

Theorem (Berthe, Brosse, H., van den Heuvel, Hoppenot, Pierron (2025))

Let G be a graph. We can determine G from its 'k-Kempe-recolouring graph' for any $k > \chi(G) + 1$.

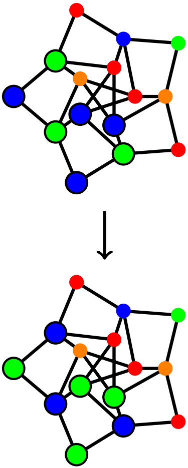

Figure: A Kempe swap.

Some Questions

Question

Given a k-recolouring graph (respectively a k-Kempe-recolouring graph), can we determine if $k = \chi(G)$?

Question

Can every graph G be determined from its $(\chi(G) + 1)$-Kempe-recolouring graph?

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What about other reconfiguration graphs?