Reconstructing a graph from its Bell colouring graph

CGO Seminar Series 2025-2026

Brian Hearn

London School of Economics, United Kingdom

13 March 2026

What is an independent set?

Definition

An independent set in a graph G is a collection of non-adjacent vertices.

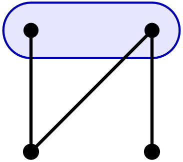


Figure: An independent set in P_4.

What is an independent set partition?

Definition

An independent set partition of a graph G is a partition of $V(G)$ into independent sets.

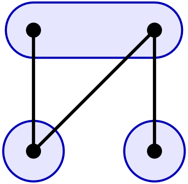


Figure: An independent set partition of P_4.

What is a Bell colouring graph?

Definition

The Bell colouring graph $\mathcal{B}(G)$ of a graph G is the graph whose vertices are the independent set partitions of G, with edges between partitions obtained from each other by changing the part of a single vertex of G.

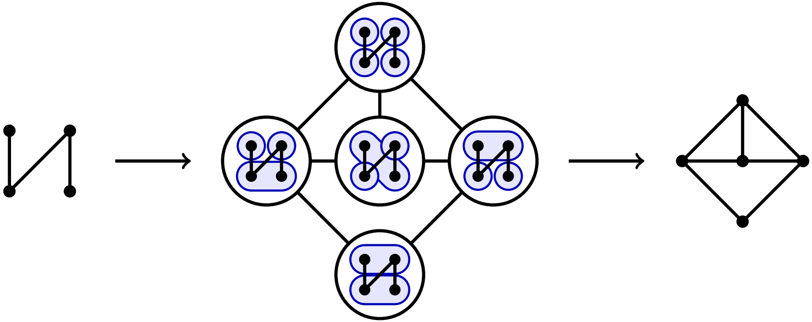


Figure: The Bell colouring graph of P_4.

What is a Bell k-colouring graph?

Definition

The Bell k-colouring graph $\mathcal{B}_k(G)$ is obtained from $\mathcal{B}(G)$ by only allowing partitions with at most k parts.

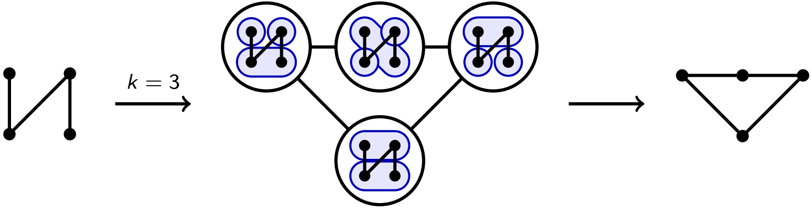


Figure: The Bell 3-colouring graph of P_4.

Why do we care?

Remark

Independent set partitions and (proper) colourings are closely related.

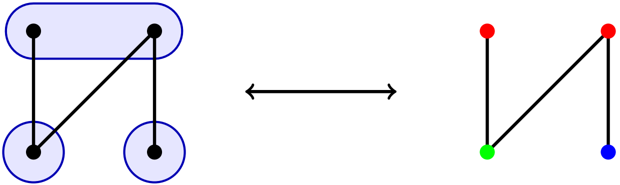


Figure: Every colouring gives an independent set partition.
Every independent set partition gives (many) colourings.

Why do we care?

Remark

Bell k-colouring graphs are closely related to k-recolouring graphs, where instead of independent set partitions, we use k-colourings.

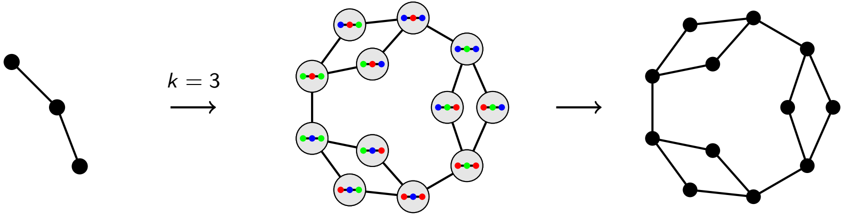


Figure: The 3-recolouring graph of P_3.

Why do we care?

Remark

Recouring graphs have been well studied, *both*

- *due to connections with computer science and theoretical physics,*
  - *see e.g. Jerrum (1995)*
- *in their own right, e.g.*
  - *Connectivity*
    - *Dyer, Flaxman, Frieze, Vigoda (2006)*
    - *Cereceda, van den Heuvel, Johnson (2008)*
  - *Hamiltonicity*
    - *Choo, MacGillivray (2011)*

Why do we care?

Remark

This has led to interest in Bell colouring graphs, *e.g.*

- *Hamiltonicity*
  - *Finbow, MacGillivray (2025+)*
- *Various structural observations*
  - *Asgarli, Krehbiel, Maclean (2025+)*

Why do we care?

Theorem (Berthe, Brosse, H., van den Heuvel, Hoppenot, Pierron, 2025+)

We can determine G from its k-recolouring graph if $k > \chi(G)$.

$\chi(G)$ is the minimum number of colours needed to colour G.

Can we reconstruct G from its Bell colouring graph?

Question

Given a Bell colouring graph $\mathcal{B}(G)$, can we reconstruct G?

Question

Given a Bell k-colouring graph $\mathcal{B}_k(G)$, can we reconstruct G?

What is already known?

Definition

The Bell colouring multigraph $\mathcal{B}^{\circ}(G)$ is obtained from $\mathcal{B}(G)$ by inserting multiple edges if there are multiple ways to move between partitions.

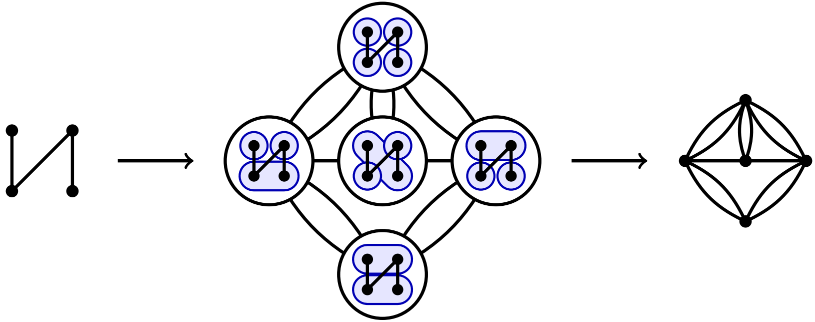


Figure: The Bell colouring multigraph of P_4.

What is already known?

Theorem (Asgarli, Krehbiel, MacLean, 2025+)

From $\mathcal{B}^{\bullet\!=\!\bullet}(G)$, we can recover G up to any universal vertices.

A vertex is universal if it is adjacent to all other vertices.

What is already known?

Theorem (Asgarli, Krehbiel, MacLean, 2025+)

From $\mathcal{B}^{\circ}(G)$, we can recover G up to any universal vertices.

A vertex is universal if it is adjacent to all other vertices.

Observation

Adding a universal vertex to G does not change $\mathcal{B}(G)$, or $\mathcal{B}^{\circ}(G)$.
(So this is best possible.)

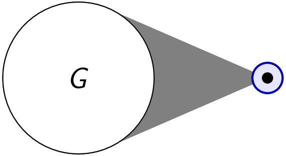


Figure: Universal vertices are always in parts of size 1.

Our results

Theorem (H., 2026++)

From $\mathcal{B}(G)$, we can recover G up to any universal vertices.

A vertex is universal if it is adjacent to all other vertices.

Our results

Theorem (H., 2026++)

From $\mathcal{B}(G)$, we can recover G up to any universal vertices.

A vertex is universal if it is adjacent to all other vertices.

Theorem (H., 2026++)

From $\mathcal{B}_k(G)$, we can recover G if $k > \chi(G)$ and $\Delta(G) < \frac{1}{9}|V(G)| - \frac{1}{3}$.

$\chi(G)$ is the least number of colours needed to colour G.

$\Delta(G)$ is the maximum degree of G.

Bell colouring vs recolouring graphs

Remark (Property of k-recolouring graphs)

Let c be a k-colouring (i.e. a vertex of a k-recolouring graph).

If not every colour is used by c, we can recolour any vertex of G.

Bell colouring vs recolouring graphs

Remark (Property of k-recolouring graphs)

Let c be a k-colouring (i.e. a vertex of a k-recolouring graph).

If not every colour is used by c, we can recolour any vertex of G.

Remark (Property of Bell colouring graphs)

Let P be a partition of $V(G)$ into independent sets.

Given a part $\{u\}$ of size 1, we may not be able to move u.

Bell colouring vs recolouring graphs

Remark (Property of k-recolouring graphs)

Each edge in a k-recolouring graph corresponds to recolouring a unique vertex of G.

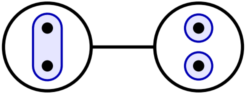
Bell colouring vs recolouring graphs

Remark (Property of k-recolouring graphs)

Each edge in a k-recolouring graph corresponds to recolouring a unique vertex of G.

Remark (Property of Bell colouring graphs)

Given a part $\{u, v\}$ of size 2, moving u into its own part is the same as moving v into its own part.



Bell colouring vs recolouring graphs

Remark (Property of k-recolouring graphs)

All edges in a clique in a k-recolouring graph correspond to recolouring the same vertex of G.

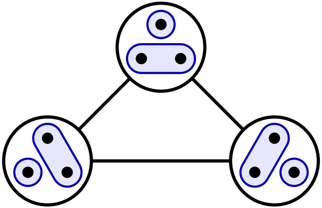
Bell colouring vs recolouring graphs

Remark (Property of k-recolouring graphs)

All edges in a clique in a k-recolouring graph correspond to recolouring the same vertex of G.

Remark (Property of Bell colouring graphs)

Given 3 independent vertices in G, we will see triangles in $\mathcal{B}(G)$ where each edge corresponds to moving a different vertex of G.



Two ways forward

Remark

Either

- *put restrictions on G to avoid small parts, or*
- *find a new strategy.*

Avoiding small parts

Theorem (H., 2026++)

From $\mathcal{B}_k(G)$, we can recover G if $k > \chi(G)$ and $\Delta(G) < \frac{1}{9}|V(G)| - \frac{1}{3}$.

$\chi(G)$ is the least number of colours needed to colour G.

$\Delta(G)$ is the maximum degree of G.

Avoiding small parts

Theorem (H., 2026++)

From $\mathcal{B}_k(G)$, we can recover G if $k > \chi(G)$ and $\Delta(G) < \frac{1}{9}|V(G)| - \frac{1}{3}$.

$\chi(G)$ is the least number of colours needed to colour G.

$\Delta(G)$ is the maximum degree of G.

Proof sketch.

Since $\Delta(G) < \frac{1}{9}|V(G)| - \frac{1}{3}$, there exists a partition $P^\dagger \in V(\mathcal{B}_k(G))$ into $\chi(G)$ parts, each of size at least 4.
Using $P^\dagger$, we can recover G using a similar approach as for the case of recolouring graphs. □

A new strategy

Theorem (H., 2026++)

From $\mathcal{B}(G)$, we can recover G up to any universal vertices.

A vertex is universal if it is adjacent to all other vertices.

A new strategy

Theorem (H., 2026++)

From $\mathcal{B}(G)$, we can recover G up to any universal vertices.

A vertex is universal if it is adjacent to all other vertices.

Idea of the proof

Let P^* be the partition of $V(G)$ into $|V(G)|$ parts of size 1.

- *Try to identify P^*.*
- *Use the local structure of $\mathcal{B}(G)$ near P^* to recover G.*

A new strategy (fixed)

Idea of the proof (fixed)

Let P^* *be the partition of* $V(G)$ *into* $|V(G)|$ *parts of size* 1*.*

- *Try to identify some* P *that looks like* P^**.*
- *Use the local structure of* $\mathcal{B}(G)$ *near* P *to recover* G*.*

What does "looks like P^*" mean?

Definition

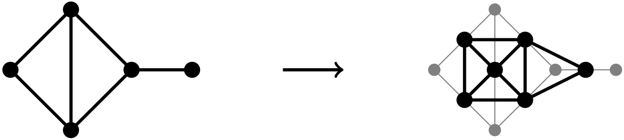
Call $P \in V(\mathcal{B})$ a P^*-candidate if

- all parts have size ≤ 3,
- "it is hard to move vertices in or out of parts of size 2 and 3".

Using P^*-candidates to determine G

Definition

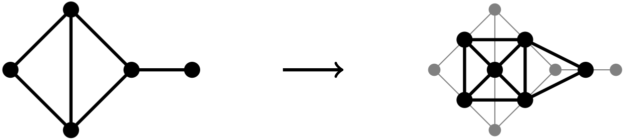
The line graph $L(G)$ of G is the graph on vertex set $E(G)$ where vertices of $L(G)$ are adjacent if and only if they are incident in G.



Using P^*-candidates to determine G

Definition

The line graph $L(G)$ of G is the graph on vertex set $E(G)$ where vertices of $L(G)$ are adjacent if and only if they are incident in G.



Theorem (Whitney, 1932)

Every connected graph is determined by its line graph, except for the triangle and the claw.

We can recover the line graph of $\overline{G}$

Lemma

The neighbourhood of any P^*-candidate is isomorphic to $L(\overline{G})$.

$\overline{G}$ denotes the complement of G.

We can recover the line graph of $\overline{G}$

Lemma

The neighbourhood of any P^*-candidate is isomorphic to $L(\overline{G})$.

$\overline{G}$ denotes the complement of G.

Proof sketch.

(For brevity, we will look only at P^* itself.)

We can recover the line graph of $\overline{G}$

Lemma

The neighbourhood of any P^*-candidate is isomorphic to $L(\overline{G})$.

$\overline{G}$ denotes the complement of G.

Proof sketch.

(For brevity, we will look only at P^* itself.)

- Neighbours of P^* correspond to non-edges of G.

We can recover the line graph of $\overline{G}$

Lemma

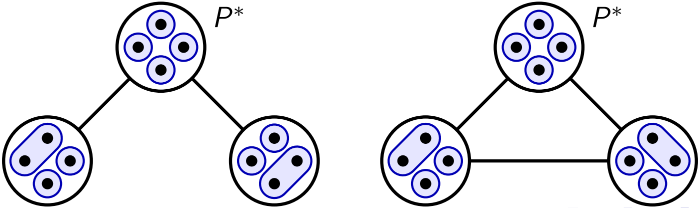
The neighbourhood of any P^*-candidate is isomorphic to $L(\overline{G})$.

$\overline{G}$ denotes the complement of G.

Proof sketch.

(For brevity, we will look only at P^* itself.)

- Neighbours of P^* correspond to non-edges of G.
- Neighbours of P^* are adjacent if and only if the corresponding non-edges are incident. □



Distinguishing between claws and triangles

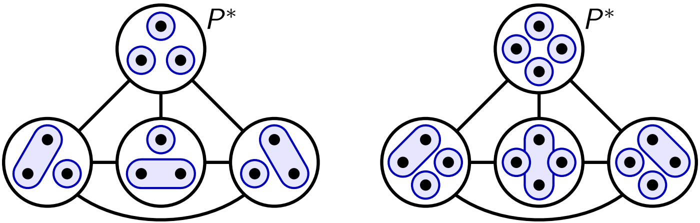
Lemma

Let P be a P^*-candidate. We can distinguish between triangles and claws in $\overline{G}$ by looking at the neighbourhood of P at distance 2.

Distinguishing between claws and triangles

Lemma

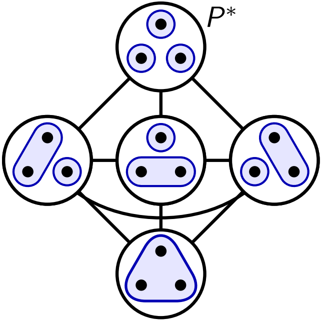
Let P be a P^*-candidate. We can distinguish between triangles and claws in $\overline{G}$ by looking at the neighbourhood of P at distance 2.

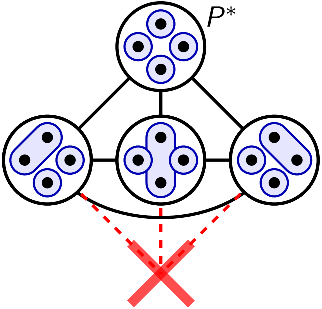


Distinguishing between claws and triangles

Lemma

Let P be a P^*-candidate. We can distinguish between triangles and claws in $\overline{G}$ by looking at the neighbourhood of P at distance 2.





Recognising P^*-candidates

Remark

If we can recognise a P^*-candidate, we are done!

Recognising P^*-candidates

Remark

If we can recognise a P^*-candidate, we are done!

Lemma

We can recognise at least one P^*-candidate.

Outline of the proof

- *List "useful" structural properties a vertex P of $\mathcal{B}(G)$ may have.*
  - *e.g. "any two non-adjacent neighbours of P have at most one common neighbour at distance 2 from P".*

Recognising P^*-candidates

Remark

If we can recognise a P^*-candidate, we are done!

Lemma

We can recognise at least one P^*-candidate.

Outline of the proof

- *List "useful" structural properties a vertex P of $\mathcal{B}(G)$ may have.*
  - *e.g. "any two non-adjacent neighbours of P have at most one common neighbour at distance 2 from P".*
- *Show that P^* satisfies these properties.*
- *Show that any vertex with maximal degree among those satisfying these properties is a P^*-candidate.*

Recognising P^*-candidates (an example)

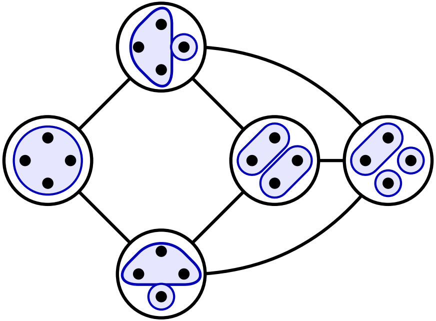
Lemma

If $P \in V(\mathcal{B}(G))$ has a part of size 4, then it has two non-adjacent neighbours with two common neighbours at distance 2 from P.

Recognising P^*-candidates (an example)

Lemma

If $P \in V(\mathcal{B}(G))$ has a part of size 4, then it has two non-adjacent neighbours with two common neighbours at distance 2 from P.



Recognising P^*-candidates

Proof sketch.

- Combining several similar lemmas to the one on the previous slide, we can find a vertex which is guaranteed to be a P^*-candidate.
- And we have already seen we can recover G (up to its universal vertices) using any P^*-candidate. □

Future research

Theorem (H., 2026++)

From $\mathcal{B}_k(G)$, we can recover G if $k > \chi(G)$ and $\Delta(G) < \frac{1}{9}|V(G)| - \frac{1}{3}$.

Future research

Theorem (H., 2026++)

From $\mathcal{B}_k(G)$, we can recover G if $k > \chi(G)$ and $\Delta(G) < \frac{1}{9}|V(G)| - \frac{1}{3}$.

Question

Can we improve the maximum degree condition to $\Delta(G) < |V(G)| - 1$?

Future research

Theorem (H., 2026++)

From $\mathcal{B}_k(G)$, we can recover G if $k > \chi(G)$ and $\Delta(G) < \frac{1}{9}|V(G)| - \frac{1}{3}$.

Question

Can we improve the maximum degree condition to $\Delta(G) < |V(G)| - 1$?

Question

What can we say if $k = \chi(G)$?

Future research

Theorem (H., 2026++)

From $\mathcal{B}_k(G)$, we can recover G if $k > \chi(G)$ and $\Delta(G) < \frac{1}{9}|V(G)| - \frac{1}{3}$.

Question

Can we improve the maximum degree condition to $\Delta(G) < |V(G)| - 1$?

Question

What can we say if $k = \chi(G)$?

Question

Can we determine if $k = \chi(G)$?