Reconstructing a graph from its Bell colouring graph

PCC 2026

Brian Hearn

London School of Economics, United Kingdom

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What is an independent set?

Definition

An independent set in a graph G is a collection of non-adjacent vertices.

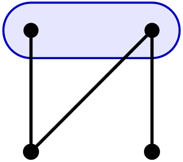

Figure: An independent set in P_4.

What is an independent set partition?

Definition

An independent set partition of a graph G is a partition of $V(G)$ into independent sets.

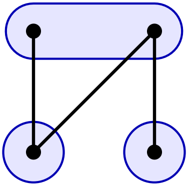

Figure: An independent set partition of P_4.

What is a Bell colouring graph?

Definition

The Bell colouring graph $\mathcal{B}(G)$ of a graph G is the graph whose vertices are the independent set partitions of G, with edges between partitions obtained from each other by changing the part of a single vertex of G.

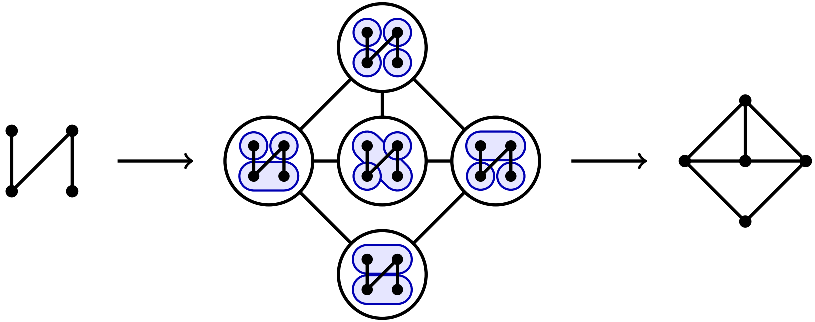

Figure: The Bell colouring graph of P_4.

Why do we care?

Remark

Independent set partitions and (proper) colourings are closely related.

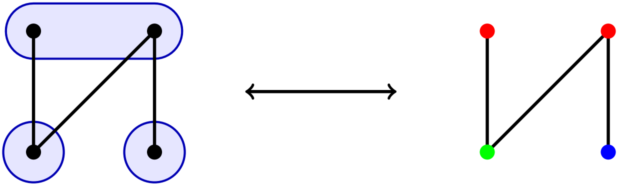

Figure: Every colouring gives an independent set partition.
Every independent set partition gives (many) colourings.

Why do we care?

Remark

Bell k-colouring graphs are closely related to k-recolouring graphs, where instead of independent set partitions, we use k-colourings.

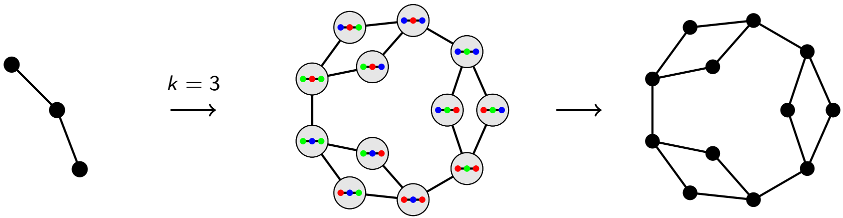

Figure: The 3-recolouring graph of P_3.

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Remark

This has led to interest in Bell colouring graphs, e.g.

- *Hamiltonicity*
  - *Finbow, MacGillivray (2025+)*
- *Various structural observations*
  - *Asgarli, Krehbiel, Maclean (2025+)*

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Theorem (Hogan, Scott, Tamitegama, Tan (2024))

We can determine G from its k-recolouring graph if $k > 5|V(G)|^2$.

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What if we only allow partitions with $\leq k$ parts?

What is already known?

Definition

The **Bell colouring multigraph** $\mathcal{B}^{\bullet=\bullet}(G)$ is obtained from $\mathcal{B}(G)$ by inserting multiple edges if there are multiple ways to move between partitions.

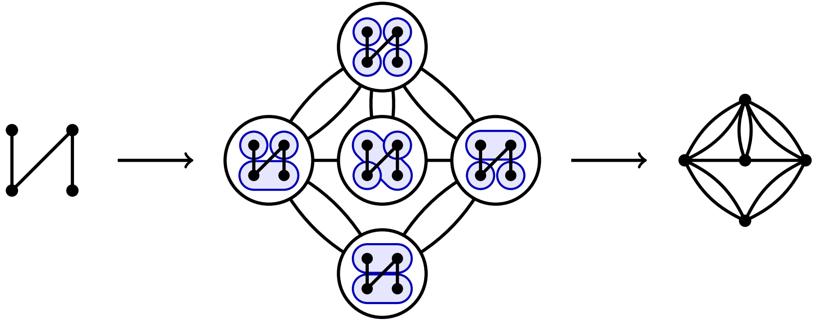

Figure: The Bell colouring multigraph of P_4.

What is already known?

Theorem (Asgarli, Krehbiel, MacLean, 2025+)

From $\mathcal{B}^{\bullet\!\circ\!\bullet}(G)$, we can recover G up to any universal vertices.

A vertex is universal if it is adjacent to all other vertices.

What is already known?

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From $\mathcal{B}^{\circ}(G)$, we can recover G up to any universal vertices.

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Observation

Adding a universal vertex to G does not change $\mathcal{B}(G)$, or $\mathcal{B}^{\circ}(G)$.
(So this is best possible.)

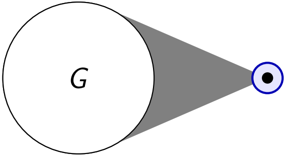

Figure: Universal vertices are always in parts of size 1.

Main result

Remark

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Our strategy

Idea of the proof

Let P^* be the partition of $V(G)$ into $|V(G)|$ parts of size 1.

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Why is finding P^* useful?

- *Neighbours of P^* correspond to non-edges of G.*

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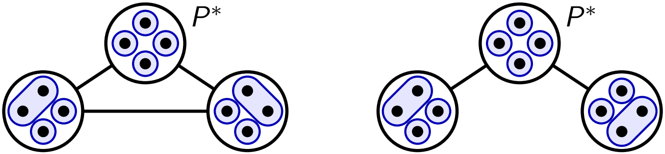
Why is finding P^* useful?

- *Neighbours of P^* correspond to non-edges of G.*
- *Neighbours of P^* are adjacent if and only if the corresponding non-edges are incident.*



Roadblocks (...versus Bell colouring multigraphs)

Remark

Without multiple edges, it's hard to find the partition P^* into $|V(G)|$ parts of size 1.

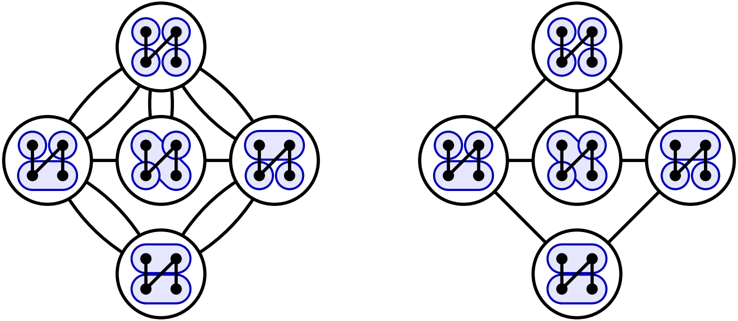


Figure: The Bell colouring multigraph of P_4 vs the Bell colouring graph of P_4.

Roadblocks (...versus recolouring graphs)

Remark

Small parts behave weirdly (compared to small colour classes).

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Example

Given a part $\{u\}$ of size 1, we may not be able to add u to any other part.

Further Results

Definition

Obtain $\mathcal{B}_k(G)$ from $\mathcal{B}(G)$ by allowing only partitions of size $\leq k$.

Obtain $\mathcal{B}_{\geq k}(G)$ from $\mathcal{B}(G)$ by allowing only partitions of size $\geq k$.

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Theorem (H., 2026+)

If $k \leq |V(G)| - 2$, we can determine G from $\mathcal{B}_{\geq k}(G)$, up to universal vertices.

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Theorem (H., 2026+)

If $k > \chi(G)$ and $\Delta(G) < \frac{1}{9}|V(G)| - \frac{1}{3}$, we can determine G from $\mathcal{B}_k(G)$.

$\chi(G)$ is the least number of colours needed to colour G.

$\Delta(G)$ is the maximum degree of G.

Future research

Theorem (H., 2026+)

From $\mathcal{B}_k(G)$, we can recover G if $k > \chi(G)$ and $\Delta(G) < \frac{1}{9}|V(G)| - \frac{1}{3}$.

Question

Can we improve the maximum degree condition to $\Delta(G) < |V(G)| - 1$?

Future research

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From $\mathcal{B}_k(G)$, we can recover G if $k > \chi(G)$ and $\Delta(G) < \frac{1}{9}|V(G)| - \frac{1}{3}$.

Question

Can we improve the maximum degree condition to $\Delta(G) < |V(G)| - 1$?

Question

What can we say if $k = \chi(G)$?

Question

Can we determine if $k = \chi(G)$?