Reconstructing Graphs from their Colouring Graphs

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What is a k-colouring graph?

Definition

Let G be a graph. Let $k \in \mathbb{N}$.

The k-colouring graph of G (denoted $\mathcal{C}_k(G)$) is the graph whose vertex set is the set of k-colourings of G, with an edge between two k-colourings if and only if they differ at exactly one vertex.

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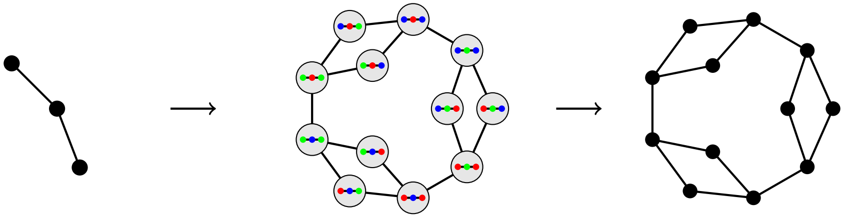
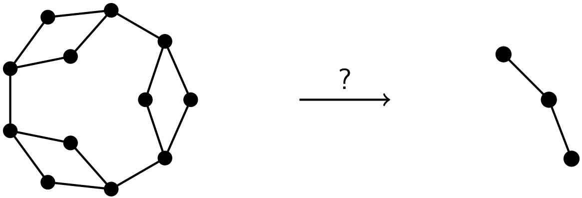


Can we reconstruct G from its k-colouring graph?

Question

Given a non-empty colouring graph $\mathcal{C}_k(G)$ (where k and G are unknown), can we reconstruct G?



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Conjecture (Asgarli, Krehbiel, Levinson, Russell (2024))

For any graph G, the collection $\{\mathcal{C}_k(G)\}_{k\in\mathbb{N}}$ uniquely determines G up to isomorphism.

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There is some function $f : \mathcal{G} \to \mathbb{N}$ (where $\mathcal{G}$ is the set of all finite graphs) such that for any graph $G \in \mathcal{G}$, the finite collection $\{\mathcal{C}_k(G)\}_{k=1}^{f(G)}$ uniquely determines G up to isomorphism.

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Theorem (Hogan, Scott, Tamitegama, Tan (2024))

Let G be a graph on n vertices. For any natural number $k > 5n^2$, the k-colouring graph $\mathcal{C}_k(G)$ uniquely determines G up to isomorphism.

Our result

Observation

$\mathcal{C}_k(G)$ is empty (has no vertices) if $k < \chi(G)$.

Recall: the chromatic number $\chi(G)$ of G is the minimum number of colours needed for a proper colouring of G.

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Remark

The proof is based on the ideas from the Hogan, Scott, Tamitegama, and Tan proof.

Some observations / Idea of the proof

We want to understand how the structure of $\mathcal{C}_k(G)$ relates to G.

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The edges of each **clique** *in $\mathcal{C}_k(G)$ all correspond to the* **same** *vertex of G.*

So each clique in $\mathcal{C}_k(G)$ corresponds to a vertex of G. But how do we know which cliques correspond to which vertices?

Some observations / Idea of the proof

Idea of the proof

If we fix a $c \in \mathcal{C}_k(G)$, **distinct** *cliques containing* c *correspond to* **distinct** *vertices of* G.

If we choose an appropriate c, *looking at common neighbours of these cliques tells us if the corresponding vertices of* G *are joined by an edge.*

The local structure of $\mathcal{C}_k(G)$

Definition

Let $c \in \mathcal{C}_k(G)$. Let

$$\mathcal{J}(c) := \{\text{maximal cliques in } \mathcal{C}_k(G) \text{ which contain } c\}.$$

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The local structure of $\mathcal{C}_k(G)$

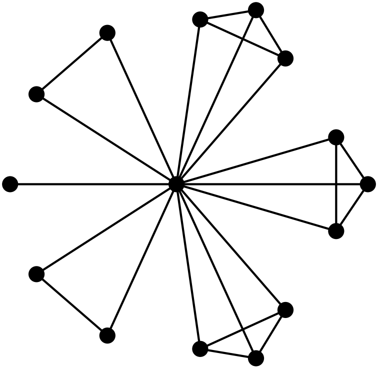

Figure: A colouring $c \in \mathcal{C}_k(G)$ and the maximal cliques in which c is contained.

$|\mathcal{J}(c)| \leq |G|$ for all $c \in \mathcal{C}_k(G)$

Observation

Given any $c \in \mathcal{C}_k(G)$, each $J \in \mathcal{J}(c)$ corresponds to **some** *vertex v of G.*

And no two distinct $J, K \in \mathcal{J}(c)$ correspond to the **same** *vertex v of G.*

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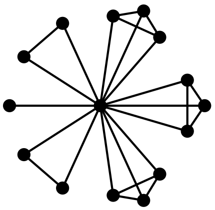

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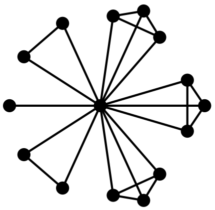

Definition

Given $v \in G$, we will denote by J_v the unique $J \in \mathcal{J}(c)$ which corresponds to v, if it exists.

Can $|\mathcal{J}(c)| = |G|$?

Observation

Let $c \in \mathcal{C}_k(G)$. If, under the colouring c, every vertex of G can be recoloured, then $|\mathcal{J}(c)| = |G|$.

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Lemma

Let $k > \chi(G)$. Then there exists $c \in \mathcal{C}_k(G)$ such that $|\mathcal{J}(c)| = |G|$.

An edge-detecting lemma

Lemma

Let $c \in \mathcal{C}_k(G)$*. For any* $u, v \in G$ *distinct such that* J_u *and* J_v *exist, if there exist* $c_u \in J_u \setminus \{c\}$ *and* $c_v \in J_v \setminus \{c\}$ *such that*

$$N(c_u) \cap N(c_v) = \{c\},$$

then $uv \in E(G)$*.*

What does this mean?

Question

Every $c \in \mathcal{C}_k(G)$ detects some **subset** *of the edges of G.*

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Every $c \in \mathcal{C}_k(G)$ *detects some* **subset** *of the edges of* G.

But does there exist $c \in \mathcal{C}_k(G)$ *which detects* **every** *edge?*

The answer is yes! (when $k > \chi(G)$)

Lemma

Let $k > \chi(G)$, and let $c \in \mathcal{C}_k(G)$ be any colouring using $\chi(G)$ colours.

If $uv \in E(G)$, then there exists some $c_u \in J_u \backslash \{c\}$ and $c_v \in J_v \backslash \{c\}$ such that $N(c_u) \cap N(c_v) = \{c\}$.

Are we done?

Observation

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End of proof.

Compute G_c for all $c \in \mathcal{C}_k(G)$. Since $k > \chi(G)$, there exists some c^* such that $G_{c^*} = G$.

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End of proof.

Compute G_c for all $c \in \mathcal{C}_k(G)$. Since $k > \chi(G)$, there exists some c^* such that $G_{c^*} = G$.

We can recognise such a G_{c^*} since it will have a maximal number of edges among all G_c (since every G_c is a subgraph of G). $\square$

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Every connected bipartite graph (on at least 2 vertices) has the same 2-colouring graph.

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Example

Every connected bipartite graph (on at least 2 vertices) has the same 2-colouring graph.

For any $\chi \geq 2$, there are infinitely many graphs of chromatic number χ which have the same χ-colouring graph.

Example

For $k \in \mathbb{N}$, every k-tree has the same $(k+1)$-colouring graph.

Can we extend this to $k = \chi(G)$?

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Question

Maybe G has a unique $\chi(G)$-colouring graph if:

- *G has no $\chi(G)$-cliques?*
- *G has no* **frozen** *vertices?*

We call a vertex frozen if it cannot be recoloured under any $\chi(G)$-colouring.

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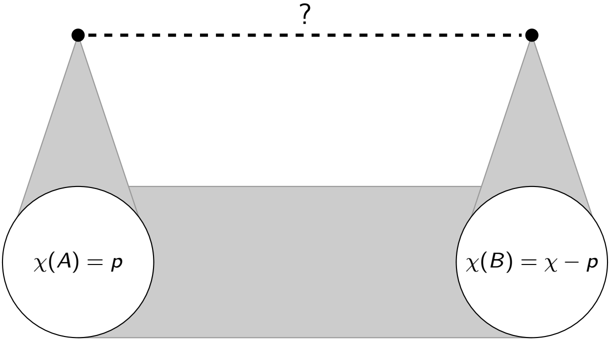
Answer

The answer (in both cases) is **not necessarily**.

A counterexample

Example

There may exist edges we cannot detect (even if G has no frozen vertices).



A counterexample

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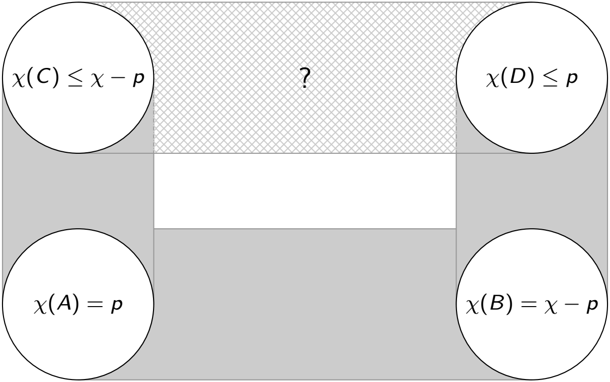
There may exist arbitrarily many edges we cannot detect (even if G has no frozen vertices).



Next steps

There are broad classes of graphs which cannot be distinguished by their $\chi-$colouring graphs.

Question

Suppose we are given some non-empty colouring graph $\mathcal{C}_k(G)$.

Can we detect if $k = \chi(G)$?