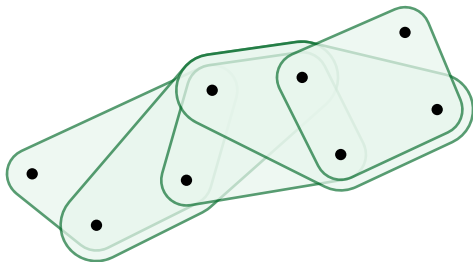


Spanning tight components in 4-uniform hypergraphs



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London School of Economics, United Kingdom

11th Polish Combinatorial Conference, Będlewo, Poland
24 September 2026

Strong connectivity in hypergraphs

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A k -uniform hypergraph (k -graph) is a hypergraph where all edges contain exactly k vertices.

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Example

- Dirac: Minimum degree $n/2 \implies$ Hamiltonicity.

Spanning subgraphs and tight connectivity

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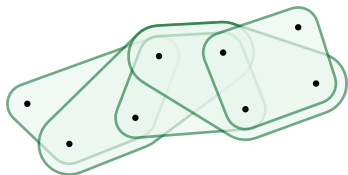


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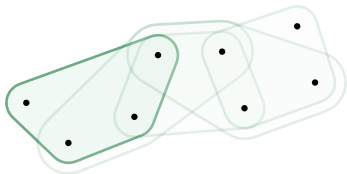


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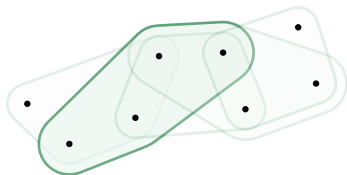


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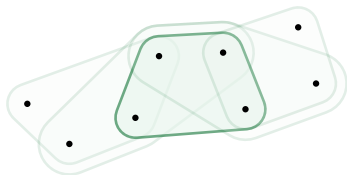


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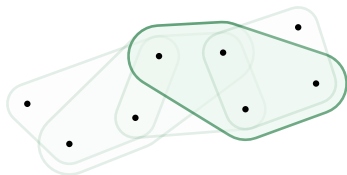


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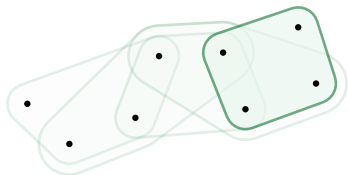


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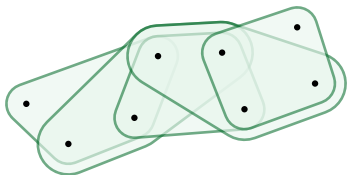


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Spanning subgraphs and tight connectivity

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- A **tight component** is an edge-maximal tightly connected subgraph.

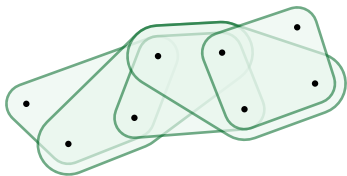


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Spheres in hypergraphs

“Definition”

A $(k - 1)$ -sphere in a k -graph is a tightly connected subgraph “homeomorphic” to a $(k - 1)$ -sphere.

Example: a spanning 1-sphere in a graph is a Hamilton cycle.

Spheres in hypergraphs

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Question (Conlon, Gowers)

What degree conditions force a spanning $(k - 1)$ -sphere in a k -graph?

Codegree

Definition

The **codegree** of a $(k - 1)$ -tuple T is the number of edges containing T .

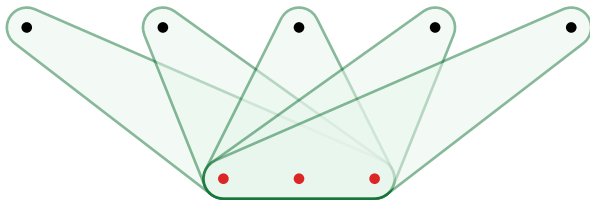


Figure: A triple of codegree 5.

Minimum codegree and spanning spheres

Theorem (Georgakopoulos, Haslegrave, Montgomery, Narayanan (2022))

Every n -vertex 3-graph with *minimum codegree* at least $n/3 + o(n)$ contains a spanning 2-sphere.

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Remark

GHMN constructed k -graphs with

- *minimum codegree* $\approx n/k$,
- no spanning $(k - 1)$ -spheres,
- no *spanning tight components*.

Minimum codegree and spanning tight components

Conjecture (Illingworth, Lang, Müyesser, Parczyk, Sgueglia (2025))

*Every n -vertex k -graph with minimum codegree at least $\lfloor n/k \rfloor$ has a **spanning tight component**.*

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Theorem (Di Braccio, H., Lada, Neve, Zhang (2026+))

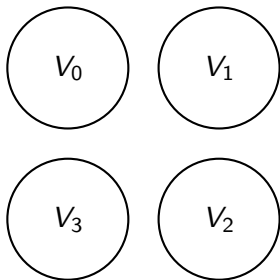
*Every n -vertex 4-graph with minimum codegree at least $\lfloor n/4 \rfloor$ has a **spanning tight component**.*

$\lfloor n/4 \rfloor$ is tight

Construction (Georgakopoulos,
Haslegrave, Montgomery, Narayanan (2022))

Equipartition $V = V_0 \sqcup V_1 \sqcup V_2 \sqcup V_3$.

For each $v \in V_i$, set $v' := i$.



$\lfloor n/4 \rfloor$ is tight

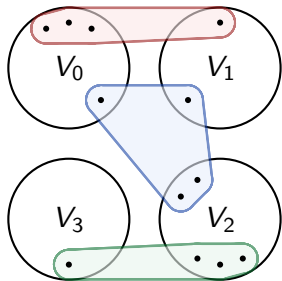
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Add edges $abcd$ such that:

- $a' + b' + c' + d' \equiv 1 \pmod{4}$



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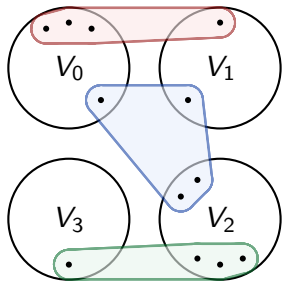
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Every tight component avoids some V_i .

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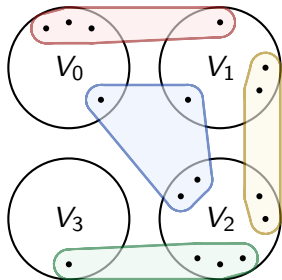
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Add edges $abcd$ such that either:

- $a' + b' + c' + d' \equiv 1 \pmod{4}$
- † $a' \equiv b' \equiv c' - 1 \equiv d' - 1 \pmod{4}$.



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Proof outline

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What's simpler than a 4-graph?

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A 3-graph!

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Preprocessing

- Each *triple* (of vertices) is contained in a unique tight component.

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(Obtain an *edge-colouring* of the complete 3-graph $K_n^{(3)}$.)

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- If any vertex misses two colours, then *merge* these colours.

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- **Main point:** Spanning colour $\implies$ Spanning tight component.
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Outline of the proof

Show that

- there are ≤ 6 colours,
- if there is no spanning tight component, then there are ≥ 7 colours.

There are ≤ 6 colours

There are ≤ 6 colours

What's simpler than a 3-graph?

There are ≤ 6 colours

What's simpler than a 3-graph?

A graph!

There are ≤ 6 colours

Lemma

Any edge-colouring of K_n where every edge lies in $\geq \lfloor (n+1)/4 \rfloor$ monochromatic triangles uses ≤ 5 colours.

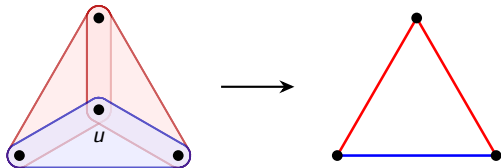
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Why is this lemma useful?

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There are ≤ 5 colours in $L(u)$.

Since u misses at most 1 colour, there are ≤ 6 colours in $K_n^{(3)}$.

Ruling out ≤ 6 colours

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What's simpler than n vertices?

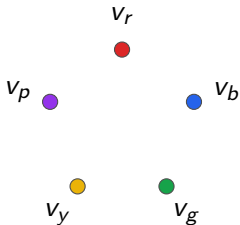
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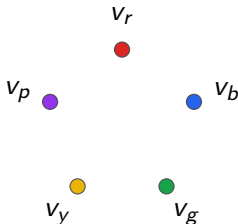
Ruling out ≤ 6 colours

For each colour c , let v_c avoid colour c .



Ruling out ≤ 6 colours

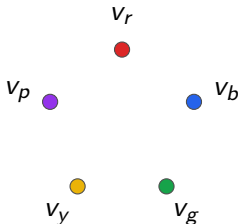
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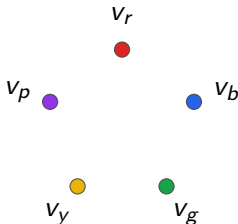


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Step 1: Understand the structure of T .

Ruling out ≤ 6 colours

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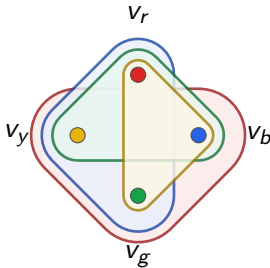
Let T be a minimal subset of $\{v_c \mid c \text{ is a colour}\}$ with no **spanning colour**.

Step 1: Understand the structure of T .

Step 2: Derive a contradiction using the minimum degree condition.

Example: ruling out 4 colours (step 1)

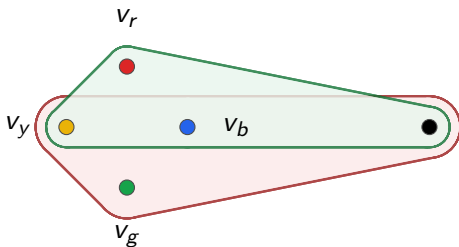
If $|T| = 4$, we get a rainbow $K_4^{(3)}$.



Example: ruling out 4 colours (step 2)

Each triple is contained in $\geq \lfloor n/4 \rfloor$ edges.

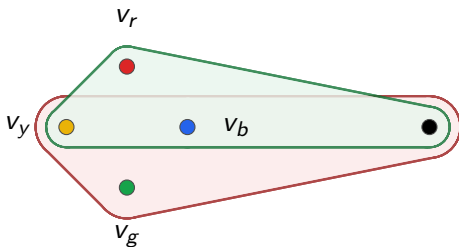
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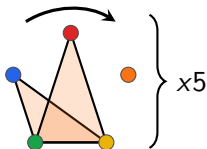


There are two tightly-connected edges with different colours. ⚡

Ruling out 5 and 6 colours

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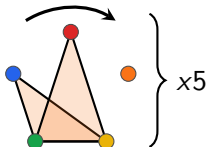
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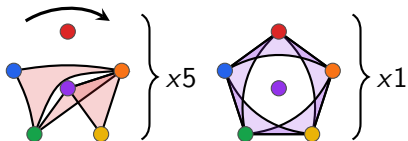
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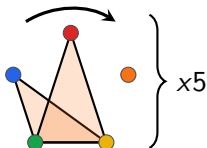
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Ruling out 5 and 6 colours

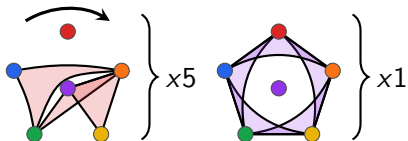
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→ ... can use these structures to derive a contradiction.



Open problems

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Every n -vertex k -graph with minimum codegree at least n/k contains a

- *spanning tight component (ILMPS, 2025),*
- *spanning $(k - 1)$ -sphere (GHMN, 2022).*

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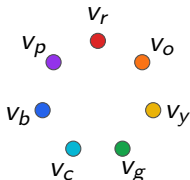
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How do “ r -configurations” behave for $r \geq 7$?

Question (from upper regime)

Can we bound the number of colours in an edge-colouring of K_n where every edge lies in at least $f(n)$ monochromatic triangles?